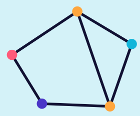
Sum+hing else

BEYOND CURRICULUM: GRAPHS

EXAMPLES

Lesson 1 - What is a graph?

8 examples | 1 challenge | Worked solutions



Try it. Try it again. Then check.

Use alongside Lesson 1 of the main Graphs mini-course.

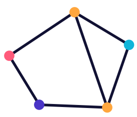
How to use this booklet

Each example is designed to be attempted more than once. The aim is not to get to the model solution quickly; it is to stay with the problem until you think you have a complete solution of your own.

Idea Learning tip

Do not check the answer after your first attempt. Try the problem again. Change your diagram, test a different idea, or explain your reasoning more carefully. Only when you genuinely think you have the right solution should you use the answer check.

1. **Try it. Then try it again.** Make a serious first attempt, then go back and have another go before checking anything.
2. **Check only the answer.** Once you think your solution is right, compare your final answer with the short answer check. Do not read the model solution yet.
3. **Go back to the problem.** If your answer differs, work out why. Even if it matches, try to improve or justify your reasoning.
4. **Then read the model solution.** Compare its reasoning with yours. Look for an idea, shortcut or explanation you can reuse next time.



You do not need to complete the examples in one sitting.

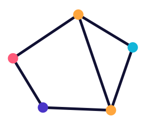
LESSON 1

What is a graph?

Extra examples: from quick checks to open-ended investigations.

Idea Learning tip

The examples are deliberately mixed. Some should feel quick; others are meant to take several attempts. If one seems too easy, try to explain your answer in a way that would convince someone else.



LESSON 1 - WHAT IS A GRAPH?

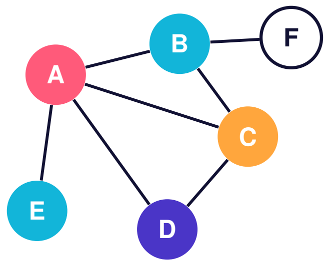
Example 1

Same graph or different graph?

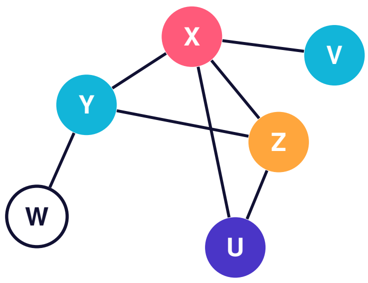
EXAMPLE

Match the vertices

The two pictures below represent graphs with six vertices. Decide whether they are really the same graph, just drawn and labelled differently.
If they are the same, match each vertex in Graph A with the corresponding vertex in Graph B.



Graph A



Graph B

Idea Learning tip

Start with something that does not depend on the drawing. Vertex degree is useful. Then look at which sorts of vertices are connected to which.

First attempt

Space for working

Try it again

Can you justify every match?

Space for working

I think I have it - check the answer →

ANSWER CHECK

Example 1

Same graph or different graph?

SHORT ANSWER ONLY

Yes. One matching is

$$A \leftrightarrow X, \quad B \leftrightarrow Y, \quad C \leftrightarrow Z, \quad D \leftrightarrow U, \quad E \leftrightarrow V, \quad F \leftrightarrow W.$$

Idea Learning tip

Do not move on to the model solution yet. Go back to your own work. If your answer is different, find the point where your reasoning changed course. If your answer matches, see whether you can make your argument clearer, shorter or more convincing.

One more attempt

Space for working

← Back to the problem

Now compare with the model solution →

MODEL SOLUTION

Example 1

Same graph or different graph?

MODEL SOLUTION

Ignore the positions of the dots

A graph is determined by its connections, not by where its vertices are drawn. We can use degrees and neighbours to identify the vertices.

Step 1 - find the unique degree-four vertex

In Graph A, only A has degree 4. In Graph B, only X has degree 4. So $A \leftrightarrow X$.

Step 2 - use the low-degree vertices

E and F are the two degree-one vertices. E is attached to A, so it must match the degree-one vertex attached to X, namely V. Thus $E \leftrightarrow V$, and therefore $F \leftrightarrow W$.

Step 3 - finish the matching

D is the only degree-two vertex, so $D \leftrightarrow U$. The two remaining degree-three vertices must be $B \leftrightarrow Y$ and $C \leftrightarrow Z$; their adjacencies agree.

Conclusion

The two drawings represent the same graph. Moving vertices, changing edge lengths and relabelling vertices do not change the underlying connection pattern.

← Back to the problem

Answer check

Next example →

LESSON 1 - WHAT IS A GRAPH?

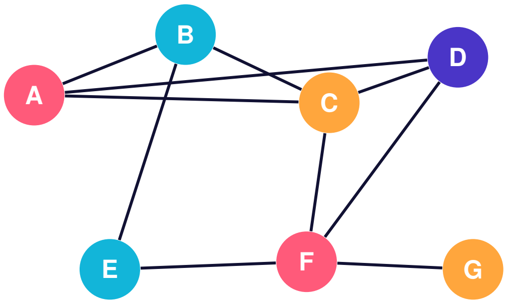
Example 2

Degree detective

EXAMPLE

Count carefully, then check yourself

Find the degree of every vertex in the graph below. Then add the seven degrees and use the degree-sum rule to check your answer.



Idea Learning tip

Do not count the same edge twice at one vertex. A useful method is to point to a vertex, trace each incident edge once, and write the degree beside the vertex before moving on.

First attempt

Space for working

Try it again

Does the degree sum equal twice the edge count?

Space for working

I think I have it - check the answer →

ANSWER CHECK

Example 2

Degree detective

SHORT ANSWER ONLY

$\deg(A) = 3$, $\deg(B) = 3$, $\deg(C) = 4$, $\deg(D) = 3$, $\deg(E) = 2$, $\deg(F) = 4$, $\deg(G) = 1$. The total is 20, matching 2×10.

Idea Learning tip

Do not move on to the model solution yet. Go back to your own work. If your answer is different, find the point where your reasoning changed course. If your answer matches, see whether you can make your argument clearer, shorter or more convincing.

One more attempt

Space for working

← Back to the problem

Now compare with the model solution →

MODEL SOLUTION

Example 2

Degree detective

MODEL SOLUTION

Count incident edges, not nearby vertices

The degree of a vertex is the number of edges that meet it. We can record the degrees one vertex at a time.

Step 1 - list the degrees

$$\deg(A) = 3, \quad \deg(B) = 3, \quad \deg(C) = 4, \quad \deg(D) = 3,$$
$$\deg(E) = 2, \quad \deg(F) = 4, \quad \deg(G) = 1.$$

Step 2 - add them

The degree sum is

$$3 + 3 + 4 + 3 + 2 + 4 + 1 = 20.$$

Step 3 - use the independent check

The graph has 10 edges. The degree-sum rule predicts

$$2 \times 10 = 20,$$

which agrees with our count.

Conclusion

The degree list is $4, 4, 3, 3, 3, 2, 1$ if written from largest to smallest, and the degree-sum check confirms the counting.

← Back to the problem

Answer check

Next example →

LESSON 1 - WHAT IS A GRAPH?

Example 3

The missing degree

EXAMPLE

One degree is hidden

A graph has seven vertices and nine edges. Six of the vertex degrees are

$$3, \quad 4, \quad 2, \quad 1, \quad 3, \quad 2.$$

What is the degree of the seventh vertex?
Try to solve it using what you know about the sum of the degrees rather than trying to draw the graph first.

Idea Learning tip

Nine edges contribute how much altogether to the degree sum? Work that out before adding the six degrees you have been given.

First attempt

Space for working

Try it again

Can you express the missing degree as an equation?

Space for working

I think I have it - check the answer →

ANSWER CHECK

Example 3

The missing degree

SHORT ANSWER ONLY

The missing degree is 3.

Idea Learning tip

Do not move on to the model solution yet. Go back to your own work. If your answer is different, find the point where your reasoning changed course. If your answer matches, see whether you can make your argument clearer, shorter or more convincing.

One more attempt

Space for working

← Back to the problem
Now compare with the model solution →

MODEL SOLUTION

Example 3

The missing degree

MODEL SOLUTION

Use the degree-sum rule

Every edge contributes 2 to the total degree, so nine edges give a total degree sum of 18.

Step 1 - add the known degrees

$$3+4+2+1+3+2=15.$$

Step 2 - find what is missing

If the missing degree is d, then

$$15+d=18.$$

Therefore

$$d=3.$$

Conclusion

The seventh vertex has degree 3. Notice that we did not need to know what the graph looked like.

← Back to the problem

Answer check

Next example →

Example 4

Could this happen?

EXAMPLE

Five vertices of degree three

Could a simple graph have exactly five vertices, with every vertex having degree 3?
Do not try to list every possible graph. Use a property you have already learned to decide whether such a graph can exist.

Idea Learning tip

What would the sum of the five degrees be? What do you know about the parity of a degree sum?

First attempt

Space for working

Try it again

Can you explain the impossibility in one sentence?

Space for working

I think I have it - check the answer →

ANSWER CHECK

Example 4

Could this happen?

SHORT ANSWER ONLY

No. Five vertices of degree 3 would have degree sum 15, but every graph has an even degree sum.

Idea Learning tip

Do not move on to the model solution yet. Go back to your own work. If your answer is different, find the point where your reasoning changed course. If your answer matches, see whether you can make your argument clearer, shorter or more convincing.

One more attempt

Space for working

← Back to the problem

Now compare with the model solution →

MODEL SOLUTION

Example 4

Could this happen?

MODEL SOLUTION

Look for a contradiction

If all five vertices had degree 3, their degrees would add to 15.

Why is that impossible?

The degree-sum rule says

$$\text{sum of degrees} = 2 \times \text{number of edges}.$$

The right-hand side is always even. But 15 is odd.

The same idea in another form

There would also be five odd-degree vertices. The Handshaking Lemma says the number of odd-degree vertices must be even.

Conclusion

No such graph exists. The contradiction appears before we draw a single edge.

← Back to the problem
Answer check
Next example →

LESSON 1 - WHAT IS A GRAPH?

Example 5

Can you build it?

EXAMPLE

Construct a graph from its degrees

Draw a simple graph on seven vertices whose degree list is

$$4, \quad 3, \quad 3, \quad 2, \quad 2, \quad 1, \quad 1.$$

Label the vertices so that someone else can check your work. Then verify the degree of every vertex.

Idea Learning tip

Start with the vertex that needs the largest degree. Each edge you add helps two vertices at once, so keep track of how many connections each vertex still needs.

First attempt

Space for working

Try it again

Can you find a different valid drawing?

Space for working

I think I have it - check the answer →

ANSWER CHECK

Example 5

Can you build it?

SHORT ANSWER ONLY

Yes. One valid edge set is

$$AB,\ AC,\ AD,\ AE,\ BC,\ BF,\ CG,\ DE.$$

This gives degrees $4, 3, 3, 2, 2, 1, 1$.

Idea Learning tip

Do not move on to the model solution yet. Go back to your own work. If your answer is different, find the point where your reasoning changed course. If your answer matches, see whether you can make your argument clearer, shorter or more convincing.

One more attempt

Space for working

← Back to the problem

Now compare with the model solution →

MODEL SOLUTION

Example 5

Can you build it?

MODEL SOLUTION

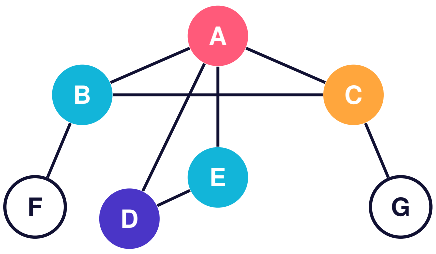
Satisfy the largest degrees first

Let A be the vertex of degree 4. Join it to four vertices, say B, C, D, E.

Step 1 - see what remains

After adding AB, AC, AD, AE, the remaining required degrees can be completed by adding

$$BC, \quad BF, \quad CG, \quad DE.$$



Step 2 - verify every vertex

The final degrees are

$$\deg(A) = 4,\ \deg(B) = 3,\ \deg(C) = 3,\ \deg(D) = 2,\ \deg(E) = 2,\ \deg(F) = 1,\ \deg(G) = 1.$$

Their sum is 16, so the graph has 8 edges, exactly as our edge list shows.

Conclusion

This is only one solution. A construction problem can have several correct graphs; what matters is that every requested degree is satisfied.

← Back to the problem   Answer check   Next example →

Example 6

Eight vertices, all degree two

EXAMPLE

Same degrees, genuinely different graphs

Draw a simple graph with exactly eight vertices in which every vertex has degree 2.
Now try again. Can you make a graph that is genuinely different - not just a redrawing or relabelling of your first graph?
Keep going. How many essentially different possibilities are there?

Idea Learning tip

If every vertex has degree 2, start following edges from any vertex. What kind of component must you eventually create?

First attempt

Space for working

Try it again

Try to explain why your list is complete.

Space for working

I think I have it - check the answer →

ANSWER CHECK

Example 6

Eight vertices, all degree two

SHORT ANSWER ONLY

There are exactly three essentially different possibilities: one 8-cycle; a 5-cycle plus a 3-cycle; or two 4-cycles.

Idea Learning tip

Do not move on to the model solution yet. Go back to your own work. If your answer is different, find the point where your reasoning changed course. If your answer matches, see whether you can make your argument clearer, shorter or more convincing.

One more attempt

Space for working

← Back to the problem

Now compare with the model solution →

MODEL SOLUTION

Example 6

Eight vertices, all degree two

MODEL SOLUTION

Every component must be a cycle

Start at any vertex and follow an edge. Because every vertex has degree 2, there is always exactly one unused way to leave a newly reached vertex. In a finite graph, the route eventually closes into a cycle.

Step 1 - simple graphs rule out tiny cycles

A cycle in a simple graph needs at least three vertices. A two-vertex cycle would require two parallel edges, which simple graphs do not allow.

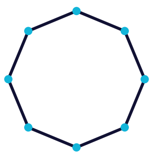
Step 2 - split eight into cycle lengths

We need to write 8 as a sum of integers, each at least 3. The only possibilities are

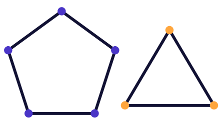
$$8, \qquad 5+3, \qquad 4+4.$$

Step 3 - interpret the three cases

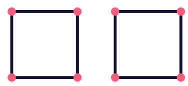
These give, respectively: one octagon; a pentagon and a triangle; or two squares. Any other drawing with all degrees 2 is just a relabelling or redrawing of one of these three structures.



8



$5+3$



$4+4$

Conclusion

There are exactly three non-isomorphic simple graphs on eight vertices in which every vertex has degree 2.

← Back to the problem
Answer check
Next example →

LESSON 1 - WHAT IS A GRAPH?

Example 7

Design a graph

EXAMPLE

Meet three conditions at once

Draw a simple graph with

- exactly six vertices,
- exactly seven edges,
- exactly four odd-degree vertices.

Then write down the degree of every vertex and check all three conditions.

Idea Learning tip

Seven edges mean the degree sum must be 14. Try inventing a six-number degree list that adds to 14 and contains exactly four odd numbers before you draw the edges.

First attempt

Space for working

Try it again

Can you make a second solution?

Space for working

I think I have it - check the answer →

ANSWER CHECK

Example 7

Design a graph

SHORT ANSWER ONLY

Many answers are possible. For example, a graph with degree list

$$3, 3, 3, 2, 2, 1$$

works: the degrees sum to 14, so there are 7 edges, and exactly four degrees are odd.

Idea Learning tip

Do not move on to the model solution yet. Go back to your own work. If your answer is different, find the point where your reasoning changed course. If your answer matches, see whether you can make your argument clearer, shorter or more convincing.

One more attempt

Space for working

← Back to the problem

Now compare with the model solution →

MODEL SOLUTION

Example 7

Design a graph

MODEL SOLUTION

Plan the degrees before the edges

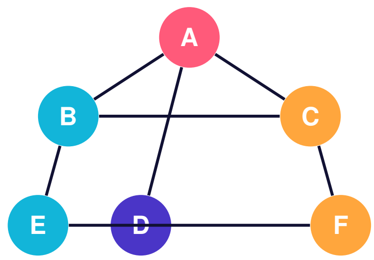
Seven edges require degree sum 14. We also want exactly four odd entries. One useful target is

$$3, 3, 3, 2, 2, 1.$$

Step 1 - one construction

Label the vertices A, B, C, D, E, F and use the edges

$$AB,\ AC,\ AD,\ BC,\ BE,\ CF,\ EF.$$



Step 2 - verify

The degrees are

$$\deg(A) = 3,\ \deg(B) = 3,\ \deg(C) = 3,\ \deg(D) = 1,\ \deg(E) = 2,\ \deg(F) = 2.$$

There are four odd degrees (A, B, C, D), and the seven listed edges are all distinct.

Conclusion

The construction works. There are other correct answers too - the conditions do not determine a unique graph.

← Back to the problem Answer check Next example →

LESSON 1 - WHAT IS A GRAPH?

Example 8

How many graphs are there?

EXAMPLE

Find every essentially different simple graph on four vertices

A **simple graph** has no loops and no repeated edges between the same pair of vertices.
Two graphs count as the **same** if one can be turned into the other just by moving the vertices, redrawing the edges, or renaming the vertices.
Draw every genuinely different simple graph on four vertices. Keep going until you think your list is complete.

Idea Learning tip

Do not check the answer after finding a few. Organise your search by the **number of edges**: zero edges, one edge, two edges, and so on. This makes duplicates and missing cases easier to spot.

First attempt - use the twelve spaces below

Continue your second attempt on the next page →

EXAMPLE 8 - SECOND ATTEMPT

Try the classification again

Remove duplicates. Look for missing edge counts. Try to convince yourself that your list is complete.

Idea Learning tip

A complete graph on four vertices has six edges. Your search only needs to go from zero to six edges.

I think I have it - check the answer →

ANSWER CHECK

Example 8

How many graphs are there?

SHORT ANSWER ONLY

There are exactly **11** essentially different simple graphs on four vertices.

Idea Learning tip

Do not move on to the model solution yet. Go back to your own work. If your answer is different, find the point where your reasoning changed course. If your answer matches, see whether you can make your argument clearer, shorter or more convincing.

One more attempt

Space for working

← Back to the problem
Now compare with the model solution →

MODEL SOLUTION

Example 8

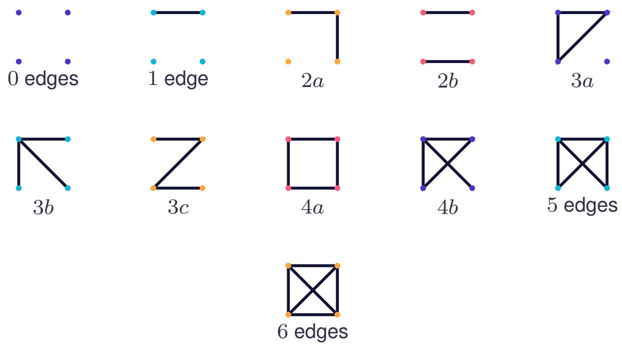
How many graphs are there?

MODEL SOLUTION

Organise by number of edges

A four-vertex simple graph can have anywhere from 0 to 6 edges. Sorting by edge count gives a systematic way to avoid duplicates and prove that the list is complete.

Edges	Number	Types
0	1	no edges
1	1	one edge
2	2	adjacent; disjoint
3	3	triangle + isolated; star; path
4	2	four-cycle; triangle + pendant
5	1	K_4 with one edge missing
6	1	complete graph K_4



Why are there only two cases with four edges?

Think of starting with the complete graph K_4, which has six edges, and deleting two edges. The two deleted edges are either adjacent or disjoint - exactly the same two structural possibilities that occur when a graph has two edges.

Add the cases

$$1 + 1 + 2 + 3 + 2 + 1 + 1 = 11.$$

Because every graph has a definite number of edges from zero to six, and we have exhausted the possibilities within each edge count, the classification is complete.

Conclusion

There are exactly 11 non-isomorphic simple graphs on four vertices. “Non-isomorphic” is the formal phrase for genuinely different connection patterns, ignoring names and drawing positions.

← Back to the problem   Answer check   Next example →

LESSON 1 CHALLENGE

Challenge

How many labelled graphs are possible?

CHALLENGE

Count without drawing them all

Suppose you have five labelled vertices

$$A, \quad B, \quad C, \quad D, \quad E.$$

We allow simple graphs only: no loops and no repeated edges.

1. How many different pairs of vertices could be joined by an edge?
2. Each possible edge can either be present or absent. How many different labelled simple graphs can you make altogether?
3. Harder: how many of those graphs have exactly four edges?

Try to solve each part without listing the graphs.

Idea Learning tip

For part 2, imagine making a yes/no choice for every possible edge. For part 3, the question changes: you must choose exactly four of the possible edges.

First attempt

Space for working

Try it again

Can you explain why your counting neither misses nor repeats a graph?

Space for working

I think I have it - check the answer →

ANSWER CHECK

Challenge

How many labelled graphs are possible?

SHORT ANSWER ONLY

There are 10 possible edges, $2^{10} = 1024$ labelled simple graphs altogether, and $\binom{10}{4} = 210$ with exactly four edges.

Idea Learning tip

Do not move on to the model solution yet. Go back to your own work. If your answer is different, find the point where your reasoning changed course. If your answer matches, see whether you can make your argument clearer, shorter or more convincing.

One more attempt

Space for working

← Back to the problem

Now compare with the model solution →

MODEL SOLUTION

Challenge

How many labelled graphs are possible?

MODEL SOLUTION

Turn graphs into choices

With labelled vertices, an edge is determined by choosing a pair of vertices. There are no loops, so the two vertices in a pair must be different.

Part 1 - count the possible edges

There are

$$\binom{5}{2} = \frac{5 \times 4}{2} = 10$$

possible pairs of vertices, hence 10 possible edges.

Part 2 - present or absent

For each of the 10 possible edges there are two independent choices: include it or do not include it. Therefore

$$2^{10} = 1024.$$

Part 3 - exactly four edges

Now we must choose exactly 4 of the 10 possible edges:

$$\binom{10}{4} = \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} = 210.$$

Conclusion

There are 10 possible edges, 1024 labelled simple graphs in total, and 210 with exactly four edges. These "how many ways?" ideas are the heart of combinatorics - the subject explored further in our Counting and Combinatorics course.

← Back to the challenge

Explore more Beyond Curriculum →