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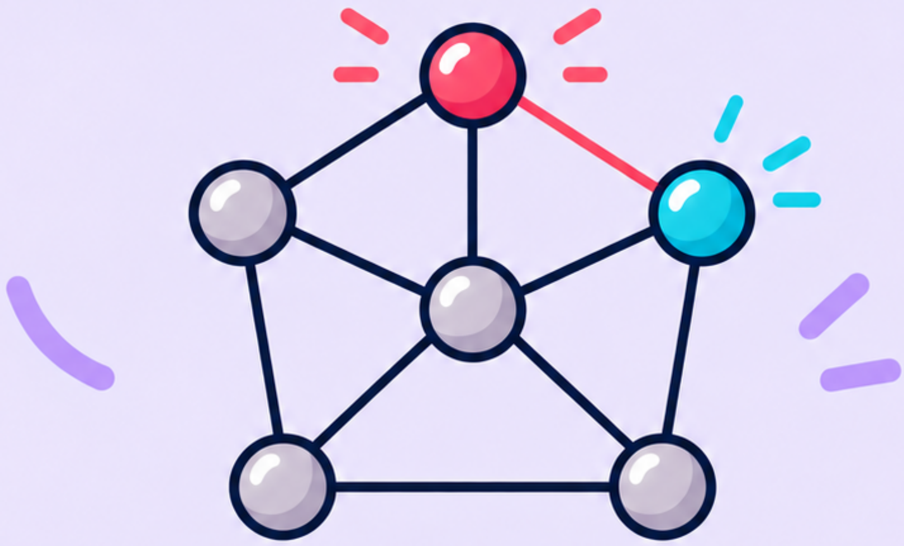
BEYOND CURRICULUM

Graphs

Bridges of Königsberg



Lesson 2

LESSON 2

Bridges of Königsberg

Can you cross every bridge exactly once?

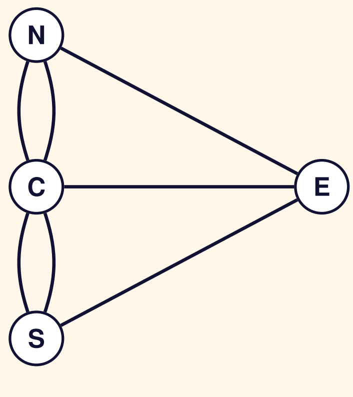
More than 300 years ago, the city of Königsberg had four areas of land connected by seven bridges. People wondered whether it was possible to take a walk that crossed **every bridge exactly once**.

Rather than reproduce the old city map here, we have already turned the bridge layout into a graph. **Each vertex represents one area of land, and each edge represents one bridge.** Two curved edges between the same pair of vertices mean there were two different bridges connecting those same areas.

TRY IT

The bridge graph

You may start anywhere and finish anywhere, but no edge may be used twice. Try to trace a route on the graph below that uses all seven edges exactly once.



Check the answers

Tip Learning tip

Do not start by guessing randomly. Choose a starting point, keep track of the edges you have used, and if your route fails, notice *where* it gets stuck.

EXAMPLE

How to read the picture

The letters N, C, S and E simply name the four land areas. The geometry of the drawing is not important. What matters is the pattern of connections — including the pairs of parallel edges, which represent different bridges between the same two areas.

TRY IT

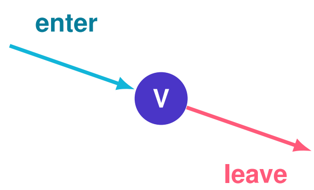
Count the degrees

What is the degree of each of the four vertices in the bridge graph? What is unusual about the four numbers?

Check the answers

Why do odd vertices cause trouble?

Whenever a route passes through a middle vertex, the edges naturally come in pairs: **one edge in + one edge out**. So every vertex that is neither the start nor the finish needs an even number of incident edges.



IMPORTANT RULE

When can every edge be used exactly once?

For a connected graph:

- **Zero odd-degree vertices**: possible, finishing where you started.
- **Two odd-degree vertices**: possible, starting at one odd vertex and finishing at the other.
- **More than two odd-degree vertices**: impossible.

The Königsberg graph has degrees $3, 5, 3, 3$. All four vertices are odd, so no route can use every bridge exactly once. The problem was not that nobody had found the right route — there was no such route to find.

Euler’s idea

In 1736, Leonhard Euler studied this problem by keeping only the information about which pieces of land were connected. This is one of the ideas that helped give rise to graph theory.

DEFINITION

Euler trail

An **Euler trail** is a route that uses every edge of a graph exactly once.

PRACTICE

Practice puzzles

Assume each graph is connected.

1. Degrees $2, 2, 2, 2$. Is an Euler trail possible? Must it finish where it started?
2. Degrees $3, 2, 2, 1$. Is an Euler trail possible? Where must it start and finish?
3. Degrees $3, 3, 3, 1$. Is an Euler trail possible?

Check the answers

CHALLENGE

Can one edge repair six odd vertices?

A connected graph has exactly six odd-degree vertices. Could adding or removing just one edge ever make an Euler trail possible? Explain.

Check the answers

Answers and discussion: Lesson 2

These are concise answers to the main questions. Open-ended drawing and design tasks usually have many possible solutions.

Bridges of Königsberg

Königsberg degrees

$3, 5, 3, 3$. All four vertices have odd degree, so no Euler trail exists.

Practice

$2, 2, 2, 2$: Euler circuit exists. $3, 2, 2, 1$: Euler trail exists, starting and finishing at the degree-3 and degree-1 vertices. $3, 3, 3, 1$: no Euler trail.

Six odd vertices

No. Adding or removing one edge changes the parity of exactly two endpoint degrees, so six odd vertices can be reduced to at best four. An Euler trail requires zero or two.