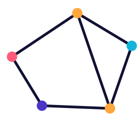
BEYOND CURRICULUM

Graphs

A printable mini-course

Ages 10+ | 6 lessons | No prerequisites



Dots, lines, routes, maps and networks

Think. Draw. Experiment. Discover.

How to use this booklet

There is no need to rush. This course is designed to be explored by **thinking, drawing, experimenting and trying things out**.

When you reach a problem, have a go before checking the answer — even if you are not sure where to start. Getting stuck for a while can be useful.

> **Learning tip**
>
> Learning tips are optional. Use them when you would like a suggestion about how to approach a problem or get more from an idea. Confident learners can simply ignore them.

When you meet a new definition, try making up your own examples — and something that is *almost* an example but fails in one important way.

> **For grown-ups**
>
> If you are working alongside a learner, try questions such as *What have you noticed?* or *What could you try next?* before offering an answer.

And do not feel you have to complete everything in one sitting. **Stopping with an interesting question still in your head is a perfectly good place to finish.**

What you will explore

1. What is a graph? — vertices, edges, degree and the Handshaking Lemma.
2. Bridges of Königsberg — when can every edge be used exactly once?
3. Paths, cycles and Euler trails — different kinds of routes through a graph.
4. One-stroke drawings — solving drawing puzzles by counting odd vertices.
5. Colouring maps — proper colourings, chromatic number and the Four Colour Theorem.
6. Networks and shortest paths — weighted graphs and Dijkstra's algorithm.

IMPORTANT RULE

One simple idea, many different problems

A graph keeps only the information about **what is connected to what**. That simple viewpoint is enough to model bridges, routes, maps, transport networks, computer networks and much more.

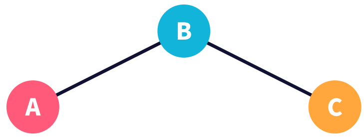
LESSON 1

What is a graph?

Start with the simplest idea: dots joined by lines.

A graph is just dots and lines

A graph is made of dots and lines. In graph theory, the dots are called **vertices** and the lines joining them are called **edges**. The exact position of the dots usually does not matter. What matters is which vertices are connected to which.



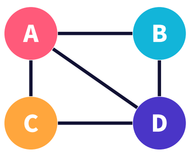
TRY IT

Quick check

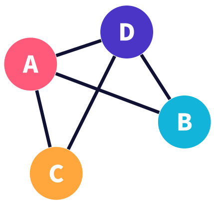
How many vertices does this graph have? How many edges?

Same graph, different picture

Moving the vertices around, changing the lengths of the edges, or even making edges cross does not usually change the graph. The pattern of connections is what matters.



Graph A



Graph B

TRY IT

Same graph?

Do Graph A and Graph B represent the same graph? Explain how you can tell.

DEFINITION

Degree of a vertex

The **degree** of a vertex is the number of edges that meet at that vertex.

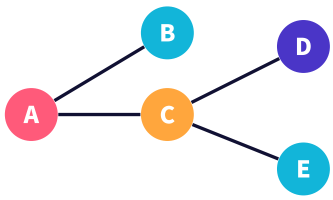
Learning tip

After reading a definition, make up one example and one non-example. An "almost right" example is especially useful: what single feature makes it fail?

The handshake puzzle

Imagine a party. Draw one vertex for each person and one edge whenever two people shake hands. Suppose the handshakes are:

Alice–Ben, Alice–Charlie, Charlie–Dom, Charlie–Ethan.



TRY IT

Count the handshakes

How many people did each person shake hands with? Add those five numbers. There were only four handshakes, so why is the total different?

IMPORTANT RULE

The degree-sum rule

$$\text{sum of the degrees} = 2 \times \text{number of edges}.$$

Every edge meets two vertices, so every edge gets counted twice when we add all the vertex degrees.

IMPORTANT RULE

Handshaking Lemma

Every graph has an **even number of vertices of odd degree**.

PRACTICE

Practice puzzles

1. Impossible party. Nine people each claim that they shook hands with exactly three other people. Could everyone be telling the truth? Explain.
2. Draw your own. Draw a graph with six vertices in which exactly four vertices have odd degree. Can you do it using exactly five edges?
3. Degree-sequence detective. Could a graph have degree list $3, 2, 2, 1, 1$? What tells you immediately?

Space for working

Learning tip

For drawing problems, try arranging the vertices around a circle first. It often makes the connections easier to see and keeps edges from accidentally passing through other vertices.

CHALLENGE

Do degrees tell you everything?

Can two different graphs have exactly the same list of vertex degrees? Draw an example or explain why not.

Space for working

LESSON 2

Bridges of Königsberg

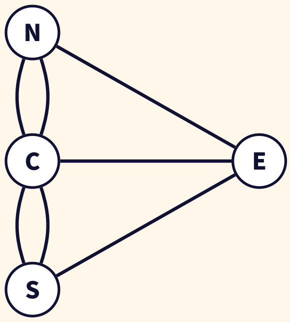
Can you cross every bridge exactly once?

More than 300 years ago, the city of Königsberg was divided by a river into four areas of land, connected by seven bridges. People wondered whether it was possible to take a walk through the city that crossed **every bridge exactly once**.

TRY IT

The bridge problem

You may start anywhere and finish anywhere, but no bridge may be crossed twice. Try to sketch a route on the graph below that uses all seven edges exactly once.



Learning tip

Do not start by guessing randomly. Choose a starting point, keep track of the edges you have used, and if your route fails, notice *where* it gets stuck.

Turn the city into a graph

The exact shapes of the islands and bridges are not important. Replace each area of land by a vertex and each bridge by an edge. The Königsberg problem becomes a graph problem.

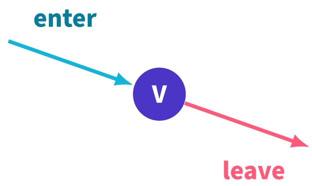
TRY IT

Count the degrees

What is the degree of each of the four vertices in the graph above? What is unusual about the four numbers?

Why do odd vertices cause trouble?

Whenever a route passes through a middle vertex, the edges naturally come in pairs: **one edge in + one edge out**. So every vertex that is neither the start nor the finish needs an even number of incident edges.



IMPORTANT RULE

When can every edge be used exactly once?

For a connected graph:

- **0 odd-degree vertices**: possible, finishing where you started.
- **2 odd-degree vertices**: possible, starting at one odd vertex and finishing at the other.
- **More than 2 odd-degree vertices**: impossible.

The Königsberg graph has degrees $3, 5, 3, 3$. All four vertices are odd, so no route can use every bridge exactly once. The problem was not that nobody had found the right route — there was no such route to find.

Euler's idea

In 1736, Leonhard Euler studied this problem by keeping only the information about which pieces of land were connected. This is one of the ideas that helped give rise to graph theory.

DEFINITION

Euler trail

An **Euler trail** is a route that uses every edge of a graph exactly once.

PRACTICE

Practice puzzles

Assume each graph is connected.

1. Degrees $2, 2, 2, 2$. Is an Euler trail possible? Must it finish where it started?
2. Degrees $3, 2, 2, 1$. Is an Euler trail possible? Where must it start and finish?
3. Degrees $3, 3, 3, 1$. Is an Euler trail possible?

CHALLENGE

Can one edge repair six odd vertices?

A connected graph has exactly six odd-degree vertices. Could adding or removing just one edge ever make an Euler trail possible? Explain.

LESSON 3

Paths, cycles and Euler trails

Learn the names for different kinds of routes through a graph.

DEFINITION

Path

A **path** follows edges without visiting the same vertex twice.

DEFINITION

Cycle

A **cycle** returns to its starting vertex without repeating any other vertex.

DEFINITION

Trail

A **trail** follows edges without using the same edge twice. It may visit the same vertex more than once.

DEFINITION

Euler circuit

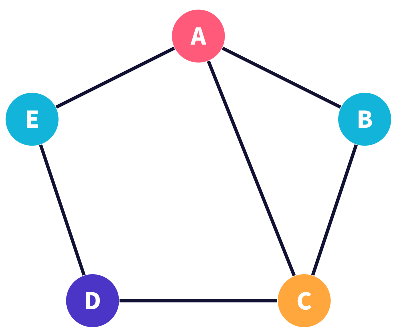
An **Euler circuit** is an Euler trail that finishes where it started.

Learning tip

For each definition, invent one example and one non-example. Ask what single feature makes the non-example fail.

What type of route is it?

Use this graph for all three routes.



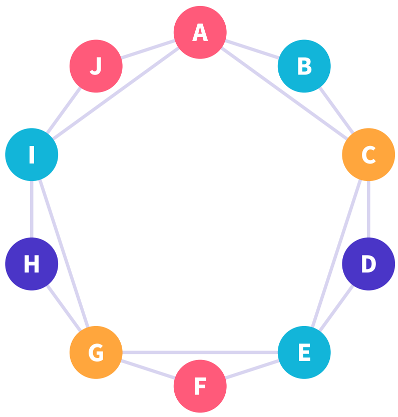
PRACTICE

Classify the routes

Route 1: $A \to B \to C \to D$
Route 2: $A \to B \to C \to D \to E \to A$
Route 3: $A \to B \to C \to D \to E \to A \to C$
For each route, decide which words apply: path, cycle, trail, Euler trail, Euler circuit.

Build a route — on paper

The online version lets you click through this graph. On paper, trace routes with a pencil or coloured pen. Do not erase failed attempts immediately — they can tell you what went wrong.



TRY IT

Four route challenges

On fresh copies of the graph if needed, try to make:

1. a path using at least four edges;
2. a cycle using at least three edges;
3. a trail that is not a path;
4. an Euler circuit using all 15 edges exactly once.

IMPORTANT RULE

The degree rule

For a connected graph:

- zero odd-degree vertices $\Rightarrow$ Euler circuit;
- two odd-degree vertices $\Rightarrow$ Euler trail but no Euler circuit;
- more than two odd-degree vertices $\Rightarrow$ no Euler trail.

PRACTICE

Use the rule without drawing

For each degree list, decide whether the graph has an Euler trail, an Euler circuit, or neither.

1. $4, 2, 2, 2, 2$
2. $4, 3, 2, 2, 1$
3. $3, 3, 2, 2, 1, 1$

CHALLENGE

Repair the graph

A connected graph has four odd-degree vertices. Add exactly one edge so that an Euler trail becomes possible. What must happen to the degrees of the two endpoints you choose?

Space for working

LESSON 4

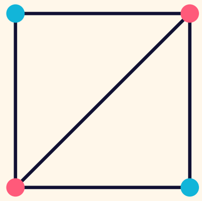
Drawing without lifting your pencil

One-stroke drawing puzzles are Euler-trail problems in disguise.

TRY IT

Can you draw it in one stroke?

Use every line exactly once without lifting your pencil.



Where should you start and finish?

> 💡 **Learning tip**
>
> Actually try the drawing. If an attempt fails, notice where you got stuck and which edges were still unused. Then try a different starting point.

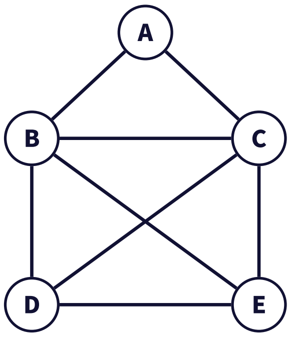
IMPORTANT RULE

When can a graph be drawn in one stroke?

Odd vertices		Conclusion
0	✓	one stroke, finish where you started
2	✓	one stroke, start and finish at the odd vertices
> 2	×	impossible to use every edge exactly once

The house

Count degrees before trying to draw.



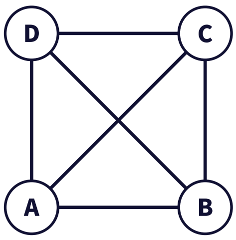
PRACTICE

Predict before drawing

Can the house be drawn in one stroke? If yes, which vertices must be the start and finish? Count the degree of each vertex first.

Does a crossing count?

Vertices are marked by dots. Two edges can cross on the page without creating a vertex.



Learning tip

Before counting degrees, decide exactly where the vertices are. A crossing is not automatically a vertex.

PRACTICE

Crossed square

Can the graph be drawn in one stroke? Explain using degrees rather than trial and error.

CHALLENGE

Fix the drawing

Add one extra edge between two corners so that the crossed square becomes drawable in one stroke. You may add a second edge between corners that are already joined; draw it as a separate curved line.

Space for working

Colouring maps

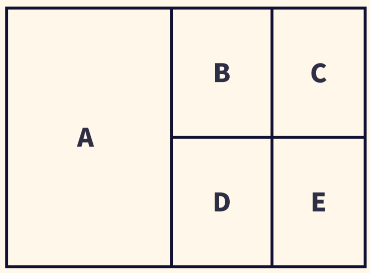
How many colours do you really need?

Maps often use different colours for neighbouring regions so that their borders are easy to see. But how many colours are really necessary?

TRY IT

A five-region map

Colour the five regions below so that any two regions sharing a side have different colours. Try to use as few colours as possible.



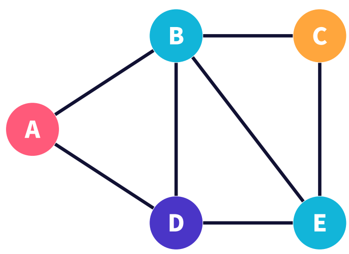
How many colours did you need? Could two colours work?

Learning tip

Do not worry about finding the best colouring immediately. Start with any sensible choice, then notice where earlier choices create problems later.

Turn a map into a graph

Replace each region by a vertex. Join two vertices when the corresponding regions share a border. The shapes and sizes disappear; the neighbour relationships remain.



DEFINITION

Proper colouring

A **proper colouring** gives colours to the vertices so that two vertices joined by an edge never have the same colour.

DEFINITION

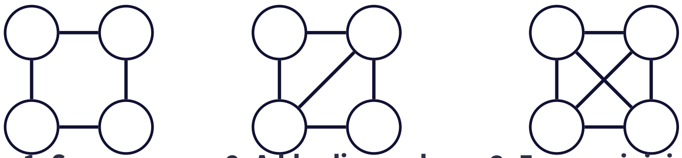
Chromatic number

The **chromatic number** of a graph is the smallest number of colours needed for a proper colouring.

PRACTICE

Find the chromatic number

Find the chromatic number of each graph.



1. Square **2. Add a diagonal** **3. Every pair joined**

IMPORTANT RULE

The Four Colour Theorem

Every flat map can be coloured using at most **four colours**, so that any two regions sharing a border have different colours.

Some maps need only two or three colours. Some genuinely need four. The statement is wonderfully simple, but proving it took mathematicians more than a century; the first accepted proof, by Kenneth Appel and Wolfgang Haken in 1976, used extensive computer calculations.

CHALLENGE

Invent a map that needs four colours

Create four regions arranged so that every region is a neighbour of each of the other three. Can you see why three colours cannot possibly work?

Space for working

LESSON 6

Networks and shortest paths

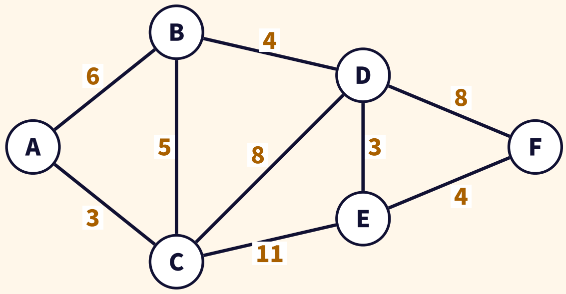
Use weighted graphs to find the quickest, cheapest or safest route.

Graphs can represent road systems, train networks, computer networks, delivery routes and many other real-world systems. Often the important question is: **what is the quickest way?**

TRY IT

Find the quickest route from A to F

The numbers are travel times. Find a route from A to F with the smallest total.



Write down at least two complete routes and compare their totals.

Learning tip

Do not just look for the cheapest next edge. A choice that looks best at the beginning may lead to a more expensive complete route.

DEFINITION

Weighted graph

A **weighted graph** is a graph in which each edge has a number, called its weight.

DEFINITION

Shortest path

The weight of a path is the sum of the weights of all its edges. A **shortest path** between two vertices is a path whose total weight is as small as possible.

Dijkstra's algorithm

Trying lots of routes works on a small graph, but large networks need a systematic method. Dijkstra's algorithm gradually discovers the shortest distance from the starting vertex to every other vertex.

IMPORTANT RULE

Dijkstra's algorithm

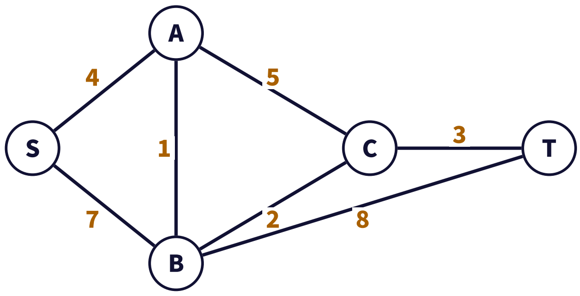
1. Give the starting vertex distance 0. Give every other vertex distance ∞.
2. Choose the unsettled vertex with the smallest distance.
3. Look at each neighbour. If going through the chosen vertex gives a shorter route, replace the neighbour's distance.
4. Mark the chosen vertex as settled and repeat.
5. Stop when the destination is settled.

Important: Dijkstra's algorithm works when edge weights are non-negative.

PRACTICE

Run Dijkstra's algorithm

Find the shortest path from S to T.



Fill in the settled order and final distance:

distance $=$

LEARNING TIP

Learning tip

Before choosing the next vertex, predict which one Dijkstra's algorithm will settle. Then check your prediction. Predicting each step is more useful than just reading a finished calculation.

Weights can mean different things

Edge weights do not have to be distances. They might represent travel time, cost, fuel use, computer delay, risk or energy use.

IMPORTANT RULE

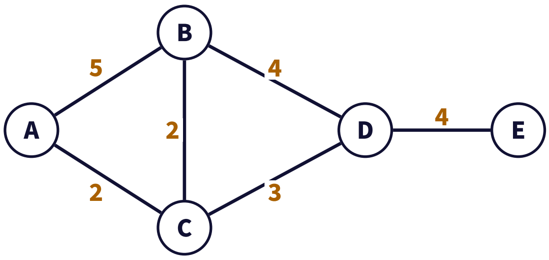
Big idea

A shortest-path problem really means: **find the route with the smallest total weight**.

PRACTICE

Three changes to a network

Start with this network:



1. Find the shortest path from A to E.
2. Now change the weight of C–D from 3 to 8. What is the new shortest path?
3. Return to the original weights, but close A–C. What is the new shortest path?

CHALLENGE

Can you trick the greedy traveller?

Design a weighted graph with start S and destination T where the **cheapest edge leaving S is not part of the shortest path**. Use at least five vertices and positive weights. Then use Dijkstra's algorithm to prove which route is shortest.

Space for working

You have reached the end of Graphs

We started with nothing more than dots and lines. Along the way, we used them to solve bridge puzzles, understand one-stroke drawings, colour maps and find routes through networks.

That is one of the remarkable things about graph theory: an extremely simple idea can describe an enormous variety of problems.

TRY IT

Where else can you spot a graph?

Think about friendships, transport maps, websites, computer networks, family trees or games. What should the vertices represent? What should the edges represent?

Space for working

Curiosity. Learning. Something more.

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Quick reference

Vertex	A dot in a graph.
Edge	A line joining two vertices.
Degree	The number of edges meeting a vertex.
Path	A route that does not repeat a vertex.
Cycle	A path that returns to its starting vertex.
Trail	A route that does not repeat an edge.
Euler trail	A trail that uses every edge exactly once.
Euler circuit	An Euler trail that returns to its starting vertex.
Proper colouring	A colouring where connected vertices have different colours.
Chromatic number	The smallest number of colours needed for a proper colouring.
Weighted graph	A graph whose edges carry numerical weights.
Shortest path	A path with the smallest possible total weight.

IMPORTANT RULE

Two rules worth remembering

Degree sum: the sum of all vertex degrees is twice the number of edges.
Euler rule: a connected graph has an Euler trail exactly when it has zero or two odd-degree vertices.

Notes and discoveries

Use this page for questions, examples you invented, graphs you want to come back to, or anything surprising you noticed.

A good mathematical question is worth keeping.

Answers and discussion

These are concise answers to the main questions. Open-ended drawing and design tasks usually have many possible solutions.

Lesson 1

Quick check

3 vertices and 2 edges.

Same graph?

Yes. Both drawings have the same four labelled vertices and exactly the same pattern of connections.

Handshake puzzle

Degrees are $2, 1, 3, 1, 1$, totalling 8. There are 4 edges. Each handshake is counted once at each endpoint, so the degree total is twice the number of handshakes.

Impossible party

No. Nine people each reporting degree 3 would give total degree $9 \times 3 = 27$, but every degree sum must be even.

Degree list $3, 2, 2, 1, 1$

Impossible: the degrees add to 9, which is odd.

Do degrees tell you everything?

No. For example, six vertices can all have degree 2 either as one six-cycle or as two separate triangles.

Lesson 2

Königsberg degrees

$3, 5, 3, 3$. All four vertices have odd degree, so no Euler trail exists.

Practice

$2, 2, 2, 2$: Euler circuit exists. $3, 2, 2, 1$: Euler trail exists, starting and finishing at the degree-3 and degree-1 vertices. $3, 3, 3, 1$: no Euler trail.

Six odd vertices

No. Adding or removing one edge changes the parity of exactly two endpoint degrees, so six odd vertices can be reduced to at best four. An Euler trail requires zero or two.

Lesson 3

Classify the routes

Route 1 is a path (and therefore also a trail), but not Euler. Route 2 is a cycle and a trail, but not an Euler circuit. Route 3 uses every edge exactly once, so it is an Euler trail; it is not a path and not an Euler circuit.

Degree lists

$4, 2, 2, 2, 2$: Euler circuit. $4, 3, 2, 2, 1$: Euler trail but no circuit; start and finish at the odd vertices. $3, 3, 2, 2, 1, 1$: four odd vertices, so neither.

Repair the graph

Join two odd-degree vertices. Their degrees each increase by 1 and become even, leaving exactly two odd vertices.

Lesson 4

First one-stroke drawing

Yes. The two endpoints of the diagonal have degree 3, while the other two vertices have degree 2. Start at one odd vertex and finish at the other.

The house

The degrees are $A = 2$, $B = 4$, $C = 4$, $D = 3$, $E = 3$. There are exactly two odd-degree vertices, D and E, so a one-stroke drawing is possible. It must start at D and finish at E, or vice versa.

Crossed square

Impossible. Each corner has degree 3. The crossing in the middle is not a vertex because there is no dot there, so there are four odd-degree vertices.

Fix the crossed square

Join any two of the four odd corners with one extra edge. Those two degrees become even, leaving exactly two odd vertices.

Lesson 5

Chromatic numbers

Square: 2. Square with one diagonal: 3, because the diagonal creates a triangle. Four vertices with every pair joined: 4.

Four-colour challenge

A correct example must have four regions where every region touches each of the other three. Equivalently, its adjacency graph is K_4.

Lesson 6

A to F

The shortest route is $A \to B \to D \to E \to F$ with total $6 + 4 + 3 + 4 = 17$.

Dijkstra practice

Settled order: S, A, B, C, T. Shortest path: $S \to A \to B \to C \to T$ with total weight $4+1+2+3 = 10$.

Network changes

In the course practice network, the original shortest path from A to E is $A \to C \to D \to E$ with total 9. If C–D rises from 3 to 8, the new shortest path is $A \to C \to B \to D \to E$ with total 12. If A–C is closed instead, the new shortest path is $A \to B \to D \to E$ with total 13.